University of Groningen
Exam Numerieke Wiskunde 1, June 20, 2014

Use of a simple calculator is allowed. All answers need to be motivated.
In front of the exercises you find its weight. In fact it gives the number of tenths which can be gained in the final mark. In total 6.0 points can be scored with this exam.

1. (a) [4] A system $LUx = b$, with L and U an $n \times n$ nonsingular lower and upper triangular matrices respectively, can be solved in two steps: $Ly = b$ and $Ux = y$. Give the according formula for y_i and x_i for $i = 1 \ldots n$.

   (b) Consider solving the linear system $Ax = b$ using an iterative method based on the splitting $A = N - P$.

      i. [2] Derive this iterative method.

      ii. [3] Also derive the recurrence relation for the error and indicate the iteration matrix.

      iii. [2] Give the necessary and sufficient requirement that must be satisfied for convergence.

   (c) i. [2] Describe the power method for the determination of the dominant eigenvalue and the corresponding eigenvector.

      ii. [4] Show that with a suitable start vector we will find this dominant eigenvalue if $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$.

      iii. [1] What determines the speed of convergence of the power method?

2. Suppose $f_1 = x_1(x_1 + x_2 - 1)$, $\quad f_2 = \frac{1}{100} + x_2 \ln(1 + x_2 - x_1)$. We want to solve $f_1(x_1, x_2) = 0$, $\ f_2(x_1, x_2) = 0$, which is written in vector form as $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. We want to solve it with a fixed point method $\mathbf{x_{n+1}} = \mathbf{g}(\mathbf{x_n})$ where $\mathbf{g}(\mathbf{x}) = \mathbf{x} + A\mathbf{f}(\mathbf{x})$.

   (a) [4] Give the Jacobian matrix of $\mathbf{f}$.

   (b) [1] Why is $(1/2, 1/2)$ a reasonable guess of the zero?

   (c) [2] From the fixed point iteration, derive the equation from which the best matrix A, based on the guess in the previous part, can be solved.

   (d) The Jacobian of $\mathbf{g}$ at the fixed point using the A of the previous part is

$$\begin{bmatrix} -0.0101028 & -0.0305152 \\ -0.0101028 & 0.0103097 \end{bmatrix}$$

Continue on other side!

i. $\boxed{1}$ Why is this matrix relevant for the study of the convergence of the fixed point method?

ii. $\boxed{2}$ Will the method converge in the neighborhood of the fixed point and why?

(e) $\boxed{2}$ Which change has to be made to the fixed point method in order to arrive at Newton's method and what is the relation to part (c).

3. (a) $\boxed{5}$ Suppose that we have an interpolating polynomial for a function $f(x)$ based on $(n+1)$ interpolation points. Now, we want to evaluate the interpolating polynomial m times (for example to plot it) with $m >> n$. How many operations need to be done to compute these m values, when Newton divided differences are used to express the interpolating polynomial? Also explain why.

(b) i. $\boxed{2}$ Give Simpson's rule for integration on the interval $[a, b]$.

ii. $\boxed{2}$ Give also the interpolating polynomial on which this rule is based expressed in Lagrange basis functions.

(c) $\boxed{5}$ Suppose we have a numerical method which approximates I by $I(h)$, where h is the mesh size and $I = I(h) + ch^3 + O(h^6)$ for some nonzero c. Derive the weighted combination of $I(h)$ and $I(2h)$ that gives a sixth-order approximation of I.

4. Consider on [0, 1] for $u(x, t)$ the diffusion equation $\partial u/\partial t = \partial^2 u/\partial x^2 + x\exp(-t)$ with initial condition $u(x, 0) = \sin(\pi x)$ and boundary conditions $u(0, t) = \sin^2(t)$ and $u(1, t) = 0$. Let the grid in x-direction be given by $x_i = i\Delta x$ where $\Delta x = 1/m$.

(a) $\boxed{3}$ Show that $\frac{\partial^2 u}{\partial x^2}(x_i, t) = \frac{u(x_{i+1},t)-2u(x_i,t)+u(x_{i-1},t)}{\Delta x^2} + O(\Delta x^2)$.

(b) $\boxed{5}$ Show that the system of ordinary differential equations (ODEs) that results from using the expression in (a) is of the form

$$\frac{d}{dt}\mathbf{u}(t) = -\frac{2}{\Delta x^2}(I - B)\mathbf{u}(t) + \mathbf{b}(t)$$

and give B, $\mathbf{b}(t)$ and $\mathbf{u}(0)$.

(c) $\boxed{2}$ Derive and sketch in the complex plane the location of the eigenvalues of B.

(d) $\boxed{2}$ Derive from the previous part that the eigenvalues of $-\frac{2}{\Delta x^2}(I-B)$ are located in the interval $[-\frac{4}{\Delta x^2}, 0]$.

(e) $\boxed{3}$ Derive the region of absolute stability of the backward Euler method.

(f) $\boxed{1}$ Use the results of the last two parts to show that there is no time step restriction if we apply the backward Euler method to the system of ODEs in part (b).

Total $\boxed{60}$